A LINEAR MODEL FOR AGGLOMERATION, DIFFUSION, AND GROWTH OF REGIONAL ECONOMIC ACTIVITY

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In a setting of regional economic development, one recognizes leading centers of activity each acting as a primum mobile in generating primary income and social and economic opportunities. These in turn stimulate and induce growth and diffusion of economic activity over the regions surrounding the leading centers. Such centers may be associated with industry, commerce or culture, the successes and failures of which tend to be dependent upon inter-regional, national, or international determinants. Thus, in terms of economic dynamics, a leading center acts as a gateway between its surrounding region and the rest of the world. It imports the fruits, economies (such as innovations and skills), and demands of the rest of the world to the benefit of the surrounding region and provides a source of external autonomous investment to the surrounding region. This source acts then as a generator or primer in stimulating and inducing regional economic activity.

In an interregional setting of economic development with full employment of productive means, the i^{th} region with gross regional product Y_i, gross regional consumption (including capital consumption) C_i, net of depreciation induced investment I_i and net of depreciation autonomous investment I_{ai} is exporting its productive means by the rate I_{ai} if $Y_i = C_i + I_i + I_{ai}$. Similarly, the j^{th} region is importing productive means by the rate I_{aj} if $Y_j = C_j + I_j - I_{aj}$. Thus, the i^{th} region is losing and the j^{th} region is gaining growth potential. One may consider autonomous investment as a mobile flow of productive means associated with political and/or economic leadership seeking for improved opportunities by relocation. These opportunities may be better in growth regions which, thereby, may become even more growth regions. In contrast, depressed regions may tend to become even more depressed. Reversals of these trends could take place when opportunities created by institutional, technological and other innovations would make a region attractive for future growth. Thus, I_{aj} benefiting the j^{th} region is associated with a set of source or base industries moving into the region j whereby this region gains growth potential not generated within its own confines.

The objective of this article is to develop a linear growth model of economic activity that illustrates the dynamics of regional economic development arising from a source created by an inflow of base industry. Such a

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source or cluster of sources pumps primary income into a region, which process in turn stimulates a diffusion and growth of induced economic activity. The magnitude of the impact of a source depends not only upon its own dynamic growth characteristics, considered autonomously given, but also upon the economic response characteristics of the surrounding region. The latter is portrayed by a linear growth process not necessarily assumed for the source or cluster of sources. The linear growth process incorporates a diffusion process generated by neighborhood contacts in the region surrounding the source. The resulting model has similarities to that suggested by Pred [10].

The linear growth and diffusion model suffices to describe the process of diffusion and growth of economic activity from a source to its surrounding regions, associating with itself an agglomeration of economic units around the source. If enough is known about the source and the response characteristics of its surrounding region, then one could separate the primary contribution from the induced contribution of gross regional product. This can be important in economic benefit analysis and in areas of economic aid, where the impact of aid could be improved by matching sources socio-culturally to the response characteristics of the regions under consideration.

Aspects of the diffusion-growth model are discussed in relation to piecewise homogeneous and non-isotropic properties of a region. An interesting combination of one dimensional diffusion-growth processes with networks of diffusion paths promises a useful application of linear graph theory to non-isotropic, non-homogeneous processes. It is pointed out that a linear model is a sufficient although not necessary starting point for a great variety of diffusion-growth models incorporating nonlinearities, non-homogeneities and non-isotropies. Some aspects of incorporating political, social, economic, and geographical barrier effects into diffusion models are also mentioned.

Measures for the Level of Regional Economic Activity

Usually, the productivity of a region is measured by its gross regional product (GRP). This is an aggregated money value per annum measure for the rate of all economic goods generated within the region at any given instant of time. For regional growth and diffusion of economic activity one introduces gross regional product density (GRPD) as a measure of economic activity for a given instant of time at a point of the region. For one dimensional problems, such as those concerned with the diffusion of economic activity along river valleys, major routes, and channels, GRPD is given with the dimensions of money value per annum per unit length. For two dimensional problems such as those concerned with diffusion processes over land areas, GRPD is given with the dimensions of money value per annum per unit area.

Let the GRP of a given region be a function of time $Y(t)$. It is the sum of final consumption $C(t)$, exports to other regions $E(t)$, and the investment in new productive means $I(t)$. The imports $M(t)$ from other regions are subtracted from Y(t) since they are not generated within the given region. One can now assume that C(t), I(t), E(t) and M(t) represent desired and planned actions of the people and decision makers in a behaviorally determined manner. Any "unwanted" effects can be given in terms of a residue flow U(t). Thus, one has the accounting identity

$$(1) \qquad Y(t) = C(t) + I(t) + E(t) - M(t) + U(t)$$

where U(t) may be either positive or negative to balance the equation.

Analogous treatment holds for GRPD. Let GRPD be represented by a function $y(x^*,t)$ where x^* is a generalized spatial position variable. Then, one has for any instant t of time and for any location x^* the consumption density $c(x^*,t)$, investment density $i(x^*,t)$, export density $e(x^*,t)$, import density $m(x^*,t)$, a residue flow $u(x^*,t)$ not accountable by the various behavioral assumptions, and the accounting identity, i.e.,

$$(2) \qquad y(x^*,t) = c(x^*,t) + i(x^*,t) + e(x^*,t) - m(x^*,t) + u(x^*,t)$$

In all the subsequent treatment it will be assumed the regional economic activity is behaviorally determined in the sense that $u(x^*,t) = 0$ for all points x^* of the region and for all $t \geq 0$.

Let dx^*_m be the infinitesimal area or length measure associated with the position variable x^* of any appropriate co-ordinate system.[1] Then, for any region R under consideration, one assumes the following relationships hold and the integrals exist:

$$(3) \qquad Y(t) = \int_R y(x^*,t)\, dx^*_m$$

$$C(t) = \int_R c(x^*,t)\, dx^*_m$$

$$I(t) = \int_R i(x^*,t)\, dx^*_m$$

$$E(t) - M(t) = \int_R [e(x^*,t) - m(x^*,t)]\, dx^*_m$$

Also, for all $t \geq 0$ one has the equations

$$(4) \qquad y(x^*,t) = c(x^*,t) + i(x^*,t) + e(x^*,t) - m(x^*,t)$$

for all x^* in R, and

$$(5) \qquad Y(t) = C(t) + I(t) + E(t) - M(t)$$

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[1]For one dimensional region $dx^*_m = dx$. For a two dimensional rectangular system $dx^*_m = dx\, dy$, and in a radially symmetric case in a polar co-ordinate system with radius r the measure is $dx^*_m = 2\pi r\, dr$.

The productive means or available factors of production (e.g., capital, labor, skills, education, managerial talent) of a region are all subject to wear and obsolescence. The portion of economic activity devoted to cancel the effects of depreciation of productive means is considered a part of final consumption. Investment activity refers to the rate of formation of new productive means (not just capital alone), net of depreciation.

Equation (4) gives GRPD in terms of the following mutually exclusive functions: $c(x^*,t)$, $i(x^*,t)$, $e(x^*,t)$ and $m(x^*,t)$. For simplicity in model building, one can choose some desired properties for these functions. For example, they may be chosen to remain bounded for all x^* of a region and for all finite values of t, with the first and second spatial derivatives existing almost everywhere and at least the first derivative with respect to time existing for all finite values of time.

GRPD values can be aggregated using a variety of sizes and shapes of sample areas. The level of aggregation will affect the spatial resolution and smoothness of the resulting sets of data points. If the sampling areas are larger than a typical average area of a neighborhood of economic activity or a trade area of a typical local economic unit, the functions can be made to vary relatively slowly over space. Such a choice assumes that the functions and their spatial derivatives do not change very much over the mean radius of a typical trade area.

Behavioral Assumptions for Regional Consumption Activity

The function $c(x^*,t)$ represents the regional consumption density at the location x^* and at the instant t of time. This depends on $y(x^*,t)$. A common but simple assumption for this relationship is

$$(6) \qquad c(x^*,t) = b\, y(x^*,t); \quad 0 \le b \le 1$$

For simplicity, the propensity to consume, b, is assumed to be homogeneous and isotropic for the region under consideration. By equations (3) and (6),

$$(7) \qquad C(t) = b\, Y(t)$$

Noting equations (6) and (7), equations (4) and (5) can be rewritten as

$$(8) \qquad (1-b)y(x^*,t) = i(x^*,t) + e(x^*,t) - m(x^*,t)$$

$$(9) \qquad (1-b)Y(t) = I(t) + E(t) - M(t)$$

The constant 1-b represents the propensity to save product from final consumption for investment in new productive means.

Behavioral Assumptions for Regional Production Activity

For the region surrounding a source it will suffice to assume a linear production function relating GRPD to $p(x^*,t)$ the value density of local productive means or the pool of factors of production,

$$(10) \qquad y(x^*,t) = (\eta/T)\; p(x^*,t)$$

η/T is a homogeneous isotropic constant by assumption, and the average rate of return on investment, the average marginal efficiency of productive means, or the average investment productivity of the local society [3, p. 73-74]. η is a dimensionless monotonic increasing function of the ratio of the total net value generated by a productive institution to the total value invested in this institution. T is the mean time of constructing a new productive institute before it becomes productive or the duration of investment in the new facility [8]. Thus, T is a characteristic reaction time of the society in converting available resources to new productive institutions and is related directly to the pay back period of a productive asset.

By definition the investment density $i(x^*,t)$ is related to $p(x^*,t)$ as

$$(11) \qquad i(x^*,t) = dp(x^*,t)/dt = (T/\eta)[dy(x^*,t)/dt]$$

noting equation (10) above. By equations (3) and (11) the following relationship between $I(t)$ and $Y(t)$ is consistent with the assumption of a linear production function,

$$(12) \qquad I(t) = (T/\eta)[dY(t)/dt]$$

In this formulation it is not necessary to assume the sources of economic activity have the same production function as the surrounding region.

Local Import-Export Density Arising From Neighborhood Effects

The flow of productive means from one region to another is associated with global import-export activity. The induction by a source of economic activity of import-export activity to its surrounding region is identified with local import-export activity. This latter process is now examined.

The flow of productive means into a region increases its production sources or reinforces the existing ones. A flow of productive means out of a region eliminates or weakens its sources. Such global movements of resources have an effect on local import-export activity in that a production source represents primary economic activity that induces secondary economic activity in the region surrounding the source. Growth pole research [2, 9] supports the contention that both primary and secondary economic activities will grow and diffuse to the region surrounding the source. An important local effect promoting such diffusion appears to be a stimulus-response chain reaction generating spatial propagation of economic activity through neighborhood contacts [5, 7, 10]. Essential to the rate of diffusion is the characteristic radius or length of a neighborhood and the characteristic reaction time of the local society. The coefficient of diffusivity is directly proportional to this radius squared and inversely proportional to the reaction time T, introduced previously.

A mechanism for the local import-export density for each point of a region is now provided.[2] A one dimensional region is assumed so that the generalized location variable x^* becomes a single spatial variable x. The frequency of neighborhood contacts of a typical economic unit is characterized by a probability density function f(z) where z is the distance from the location of the economic unit to a particular point of contact. Thus, f(z) portrays the neighborhood of economic activity for a "typical" economic unit. The mean or average radius of a typical neighborhood of economic activity for an average economic unit is

$$(13) \qquad L \cdot = \int_0^\infty z\, f(z)\, dz$$

Consider now the demand for imports at a location x, d(x,t). In Figure 1 an economic unit is located at x_o, and its frequency of contacts toward higher levels of GRPD is characterized by $f(z) = f(x_o - x)$. Psychologically, the frequency of contacts with higher levels of GRPD generate expectations and a demand for GRP by the economic unit. It is assumed that the spatial rate of change of y(x,t) varies little over an interval of region within which f(z) has most of its weight, i.e. that in this interval

$$\partial y(x,t)/\partial x \Big|_{x=x_o} \cong \frac{y(x_o,t) - y(x,t)}{x_o - x}$$

The demand for imports at location x_o, $d(x_o,t)$, is assumed to be stimulated by the difference between its level of GRPD and the higher levels of its neighbors. But these differences are weighed by f(z). Therefore,

$$\begin{aligned} d(x_o,t) &= \int_0^\infty [y(x,t) - y(x_o,t)]\, f(x_o - x)\, d(x_o - x) \\ &= -\int_0^\infty \frac{y(x_o,t) - y(x,t)}{x_o - x}\,(x_o - x)\, f(x_o - x)\, d(x_o - x) \\ &= -\partial y(x,t)/\partial x \Big|_{x=x_o} \int_0^\infty z\, f(z)\, dz \end{aligned}$$

x_o can be any interior point of the region. Noting equation (13)

$$(14) \qquad d(x,t) = -L\, [\partial y(x,t)/\partial x]$$

This demand represents a competitive opportunity to enterprises in the neighboring locations that tends to favor those from the direction of higher levels of GRPD. Thus, it is assumed imports are supplied to meet the demand, i.e.

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[2] There is much empirical evidence relating to the interaction between a hinterland and its center. Examples include the work of Berry, Barnum, and Tennant [1], Foley [4], Huff [6], and Thomas, Mitchell and Blome [11].

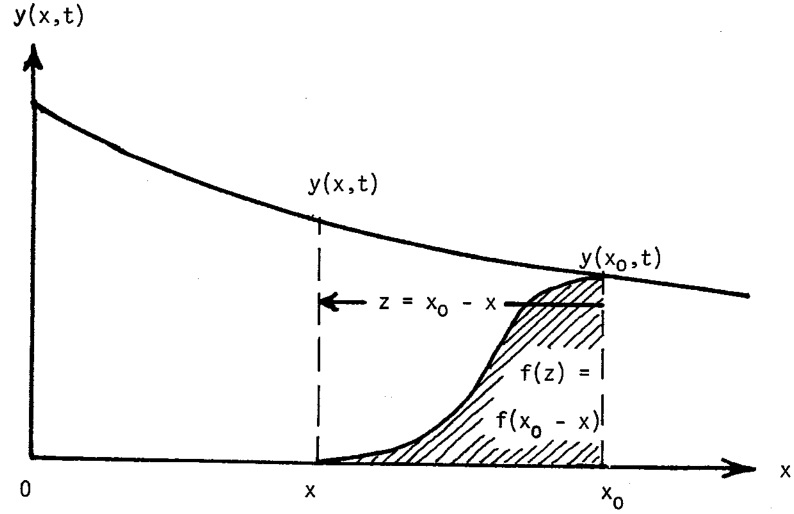


FIGURE 1

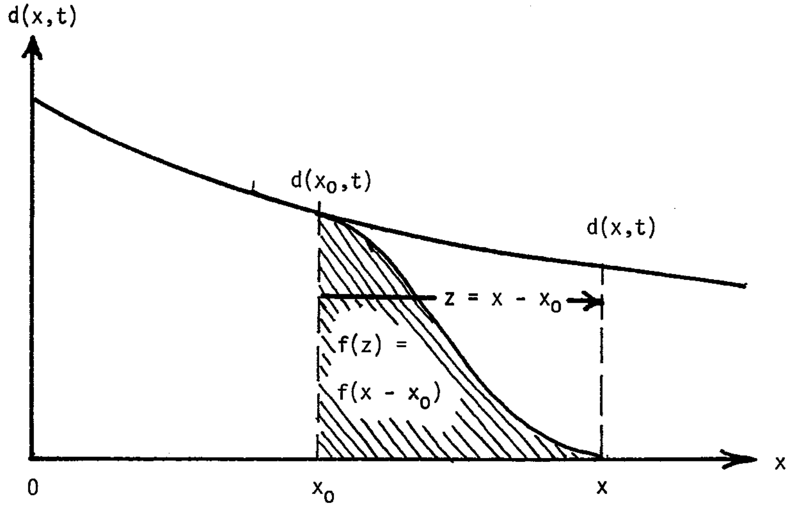


FIGURE 2

(15) $m(x,t) = d(x,t)$

The next task is to caricature the nature of GRP export density for each point x of the region. Enterprising economic units look for business opportunities where potential or actual demands may be located. Stimuli for local business and proprietor decisions are seen to originate within the neighborhood of economic activity by social contacts. Figure 2 illustrates GRP export density as a response to the GRPD demand d(x,t) weighed over f(z) characterizing the economic neighborhood. It is assumed that

$$\partial d(x,t)/\partial x\Big|_{x=x_o} \cong \frac{d(x,t) - d(x_o,t)}{x_o - x}$$

over the interval of distances where f(z) has most of its weight, and that export activity is oriented in the direction of decreasing level of GRPD. Then the expression for the GRP export density at a location x_o is

$$\begin{aligned} e(x_o,t) &= \int_0^\infty d(x,t)\; f(x-x_o)\; d(x-x_o) \\ &= \int_0^\infty [d(x,t) - d(x_o,t) + d(x_o,t)]\; f(x-x_o)\; d(x-x_o) \\ &= \int_0^\infty \frac{d(x,t) - d(x_o,t)}{x_o - x}\;(x-x_o)\; f(x-x_o)\; d(x-x_o) \\ &\qquad + d(x_o,t) \int_0^\infty f(z)\; dz \\ &= \partial d(x,t)/\partial x\Big|_{x=x_o} \int_0^\infty z\; f(z)\; dz + d(x_o,t) \end{aligned}$$

Noting this result and equations (13) and (15), the exports minus imports density at any location x is

(16) $e(x,t) - m(x,t) = L\;[\partial d(x,t)/\partial x]$

Finally, employing equation (14), assumes the following important form:

(17) $e(x,t) - m(x,t) = -L^2\;[\partial^2 y(x,t)/\partial x^2]$

Nature of Initiating Sources for Regional Economic Development

A source of regional economic activity is seen here to be a spatially confined initiating or priming activity that induces regional economic activity in its own and the surrounding region. With the source is associated a source industry (base industry) which acts as a more or less autonomous agent or gateway between the region and the rest of the world. The success or failure of source industries depends on interregional determinants so that from the point of view of the surrounding region such an industry is exogenously controlled to a predominant degree. When a source industry enters a region it brings productive means for its operation. These are combined within the region into a neighborhood complex of the primary industry.

The economic neighborhood of the source or primary industry can be viewed as follows. Consider the spatial distribution of the pool of productive means of the source or base industry (e.g., the residential distribution of labor and management personnel). This pool of productive means obtains primary income for the factors of production it provides for the source industry, and it in turn distributes a portion of this income for economic goods obtained from the surrounding region. It is the spatial distribution of the latter process which defines _the economic neighborhood of the source_ industry.

Let $h(x^*,t)$ be the probability density function representing the distribution of primary income by a source over its neighborhood at a time t. Although in general the economic neighborhoods of sources are not symmetric, for the purposes of simplicity they will be treated as such in the subsequent discussion.

As an illustration consider a two dimensional symmetric source at the location $r = 0$. Let $h(r,t)$ be its probability density function at the time t. Let $Q(t)$ be the total dollars per annum the receivers of the primary income allocate for local purchases with the above probability distribution. An average representation for a source could be chosen as follows: $y_s(0,t) = Q(t)/\pi L_s(t)^2$ where $L_s(t) = \int_0^\infty r\, h(r,t)\, 2\pi r\, dr$. The dimensions of a source are those of GRPD.

It is important to note that it has not been assumed that the economic behavioral characteristics of a source are the same as those of its surrounding region. Indeed, generally they would be different.

A GRP Growth Model: Exported versus Imported Autonomous Investment

Under the previously mentioned behavioral assumptions and by equations (5), (7) and (12) the growth equation for GRP for a particular region is

$$(18) \quad \begin{cases} dY(t)/dt - A\,Y(t) = (\eta/T)\,[M(t) - E(t)] \\ Y(0) = Y_o \end{cases}$$

where

$$(19) \quad A = (\eta/T)(1-b)$$

is the characteristic or intrinsic rate of growth of this particular region. For $M(t) - E(t) = 0$ the homogeneous solution for this growth equation is

$$(20) \quad Y(t) = Y_o\, e^{At}$$

It is not to be expected that different regions have similar growth characteristics and this assumption is not necessary, unless otherwise stated, in the subsequent discussion.

If in equation (18) $E(t) - M(t) = I_a(t)$, then the region is losing its productive means by the rate $I_a(t)$. If $E(t) - M(t) = -I_a(t)$ it would gain these means by the rate $I_a(t)$. $I_a(t)$ is the exogenously given autonomous investment.

A Special Case of Global Import-Export Dynamics

Consider two regions R_1 and R_2 each having identical consumption and production characteristics, but the former losing its growth potential while the latter gains. Table 1 gives five illustrations of $I_a(t) = E_1(t) - M_1(t) = M_2(t) - E_2(t)$ flowing from region R_1 to region R_2. The equations in this table are valid until $Y_1(t)$ of R_1 becomes zero. In such a domain $Y_1(t) + Y_2(t) = 2Y_o \exp(At)$. A pure multiplier analysis would not suffice to indicate such dynamic effects upon the growths or decays of economic activities in the regions treated.

Local Regional Economic Dynamics: A Growth-Diffusion Equation for GRPD

With the Laplacian operator ∇^2 equation (17) becomes

(17') $$e(x^*,t) - m(x^*,t) = -L^2 \nabla^2 y(x^*,t)$$

and, in addition to equation (19) defining the intrinsic rate of growth, introduce the diffusivity of GRPD as

(21) $$D = (\eta/T) L^2$$

The equations (4), (6), (11), (17'), (19), and (21) now can be combined in the growth-diffusion equation

(22) $$D \nabla^2 y(x^*,t) = [\partial y(x^*,t)/\partial t] - A\, y(x^*,t)$$

Two spatial boundary conditions and one initial condition are required for a specific solution of equation (22). For the fractional GRPD totally associated with a given source, an appropriate initial condition in the region surrounding the source is

(23) $$y(x^*,0) = 0$$

For the case of a symmetric source at the origin, an appropriate spatial boundary condition is

(24) $$y(0,t) = y_s(0,t)$$

$y_s(0,t)$ is an exogenously given source whose economic behavioral character-

TABLE I

$I_a(t)$	Y(t) of Region R_1 when $I_a(t)$ Exported from R_1 into R_2	Y(t) of Region R_2 when $I_a(t)$ Imported from R_1 into R_2
q_o (t)	$[Y_o - (\eta/T)q_o]$ exp At	$[Y_o + (\eta/T)q_o]$ exp At
I_o	$[Y_o - (I_o/(1-b))]$ exp At + $I_o/(1-b)$	$[Y_o + (I_o/(1-b))]$ exp At - $I_o/(1-b)$
$I_o$exp αt	$Y_o$exp At + $(\eta I_o/T(A-\alpha))$[exp αt - exp At]	$Y_o$exp At - $(\eta I_o/T(A-\alpha))$[exp αt - exp At]
$I_o$exp At	$[Y_o - (\eta I_o/T)t]$ exp At	$[Y_o + (\eta I_o/T)t]$ exp At
$I_o$exp 2At	$[Y_o + (I_o/(1-b))]$exp At - $(I_o/(1-b))$exp 2At	$[Y_o - (I_o/(1-b))]$exp At + $(I_o/(1-b))$exp 2At

istics are not necessarily the same as those of the surrounding region. A second spatial requirement is that $y(x^*,t)$ remains bounded for all x^* and for any finite $t \geq 0$.

Taking the Laplace transform with respect to the time of equation (22), for a one dimensional region the problem reduces to the following ordinary differential equation:

$$(25) \qquad \begin{cases} d\mathcal{Y}(x,s)/d\, x^2 - [(s-A)/D]\ y(x,s) = 0 \\ \mathcal{Y}_s(0,s) = y(0,s) \\ \mathcal{Y}(x,s) \text{ bounded for all } x \end{cases}$$

$\mathcal{Y}(x,s)$ and $\mathcal{Y}_s(0,s)$ are the Laplace transforms of $y(x,t)$ and $y_s(0,t)$, respectively. s is the complex variable of the transform. The proper solution to equation (25) is

$$(26) \qquad \mathcal{Y}(x,s) = \mathcal{Y}_s(0,s)\ \exp\left[-\sqrt{[(s-A)/D]\ x^2}\right]$$

$\mathcal{Y}_s(0,s)$ is the input and $\mathcal{Y}(x,s)$ is the regional output. Therefore, the regional transfer function is

$$(27) \qquad \mathcal{H}(x,s) = \exp\left[-\sqrt{[(s-A)/D]\ x^2}\right]$$

The simple graphical representation for the growth-diffusion input-output model is

$$\mathcal{Y}_s(0,s) \longrightarrow \boxed{\mathcal{H}(x,s)} \longrightarrow \mathcal{Y}(x,s)$$

$y(x,t)$ is obtained by taking the inverse Laplace transform of $\mathcal{Y}(x,s)$:

$$(28) \qquad y(x,t) = \mathcal{L}^{-1}\ [\mathcal{Y}(x,s)]$$

Table 2 gives four illustrations of the above, each representing a type of source. Figure 3 illustrates the input to the regional system over time of each of the four types of source. Figures 4 through 7 illustrate the respective responses $y(x,t)$. The intrinsic growth rate for all these cases is 4% per annum. The response time T is one year. x or r are given in relative units normalized by L. Thus, the spatial distances are in relative units, one unit being equal to the magnitude of L.

Figure 4 illustrates how an impulse excited "boom wave" propagates and spreads over space and time. The possibility of such boom waves of GRPD adds an important structural aspect to the theories of regional business cycles. Figures 5 and 6 illustrate how two different dynamic sources generate two different patterns of GRPD, respectively. Figure 7 illustrates the effect of a source that first declines and then grows.

For a radially symmetric source in an infinite, homogeneous, and isotropic plane, the growth-diffusion equation takes a cylindrical (Bessel) form:

TABLE 2

SOURCE $y_s(0,t)$	LAPLACE TRANSFORM OF SOURCE	LAPLACE TRANSFORM OF GRPD	GRPD $y(x,t)$
$q_o \delta(t)$	q_o	$q_o \exp[-\sqrt{(s-A)(x^2/D)}]$	$q_o \sqrt{x^2/4D\pi t^3} \exp[At-(x^2/4Dt)]$
$y_o \exp At$	$y_o/(s-A)$	$y_o \frac{\exp[-\sqrt{(s-A)(x^2/D)}]}{(s-A)}$	$y_o[\exp At] \operatorname{erfc} \sqrt{x^2/4Dt}$
$y_o\sqrt{4t/\pi T_o} \exp At$	$\frac{y_o}{\sqrt{T_o}} (s-A)^{-3/2}$	$\frac{y_o}{\sqrt{T_o}} \frac{\exp[-\sqrt{(s-A)(x^2/D)}]}{(s-A)^{3/2}}$	$y_o\sqrt{4t/\pi T_o} \exp[At-(x^2/4Dt)]$
$y_o e^{(a+A)} \operatorname{erfc}\sqrt{at}$	$\frac{y_o(s-A)^{-1/2}}{\sqrt{a}+\sqrt{s-A}}$	$\frac{y_o \exp[-\sqrt{(s-A)(x^2/D)}]}{\sqrt{s-A}[\sqrt{a}+\sqrt{s-A}]}$	$y_o \exp[(a+A)t+\sqrt{ax^2/D}]\cdot \operatorname{erfc}[\sqrt{at}+\sqrt{x^2/4Dt}]$

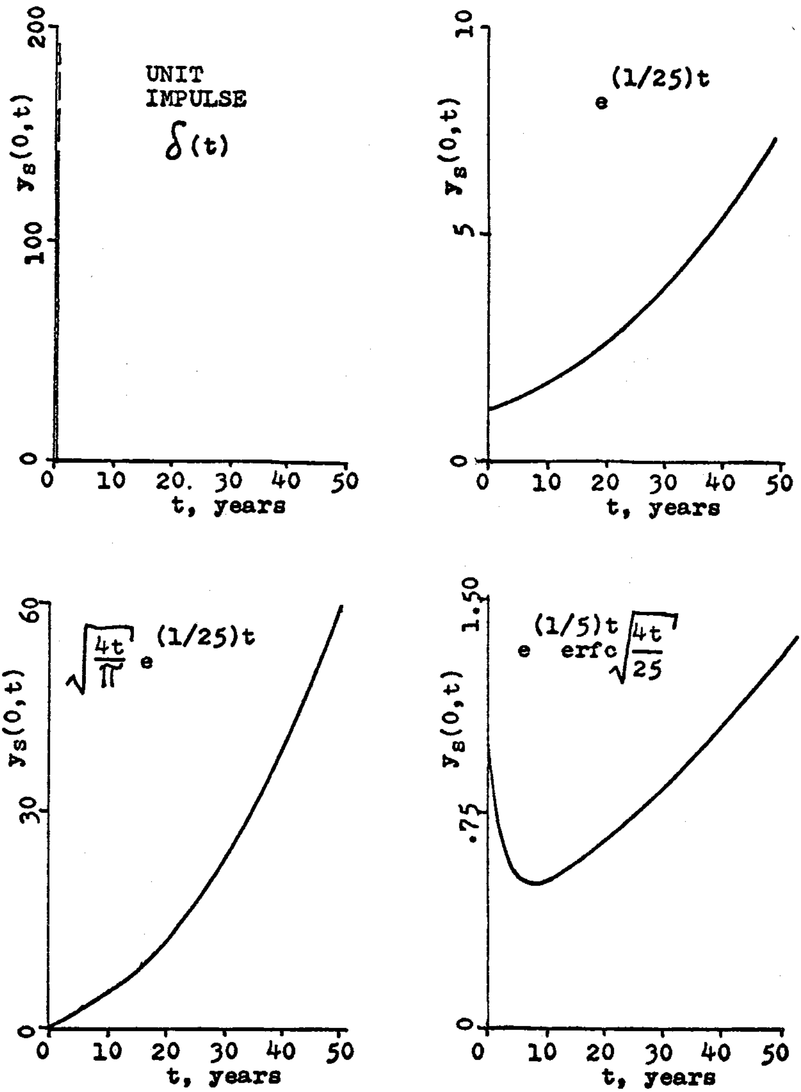

UNIT
IMPULSE
δ(t)
$e^{(1/25)t}$
$\sqrt{\frac{4t}{\pi}}\, e^{(1/25)t}$
$e^{(1/5)t}\, \mathrm{erfc}\sqrt{\frac{4t}{25}}$
$y_s(0,t)$
t, years
0
10
20.
30
40
50
100
200
5
1.50
.75
60

FIGURE 3

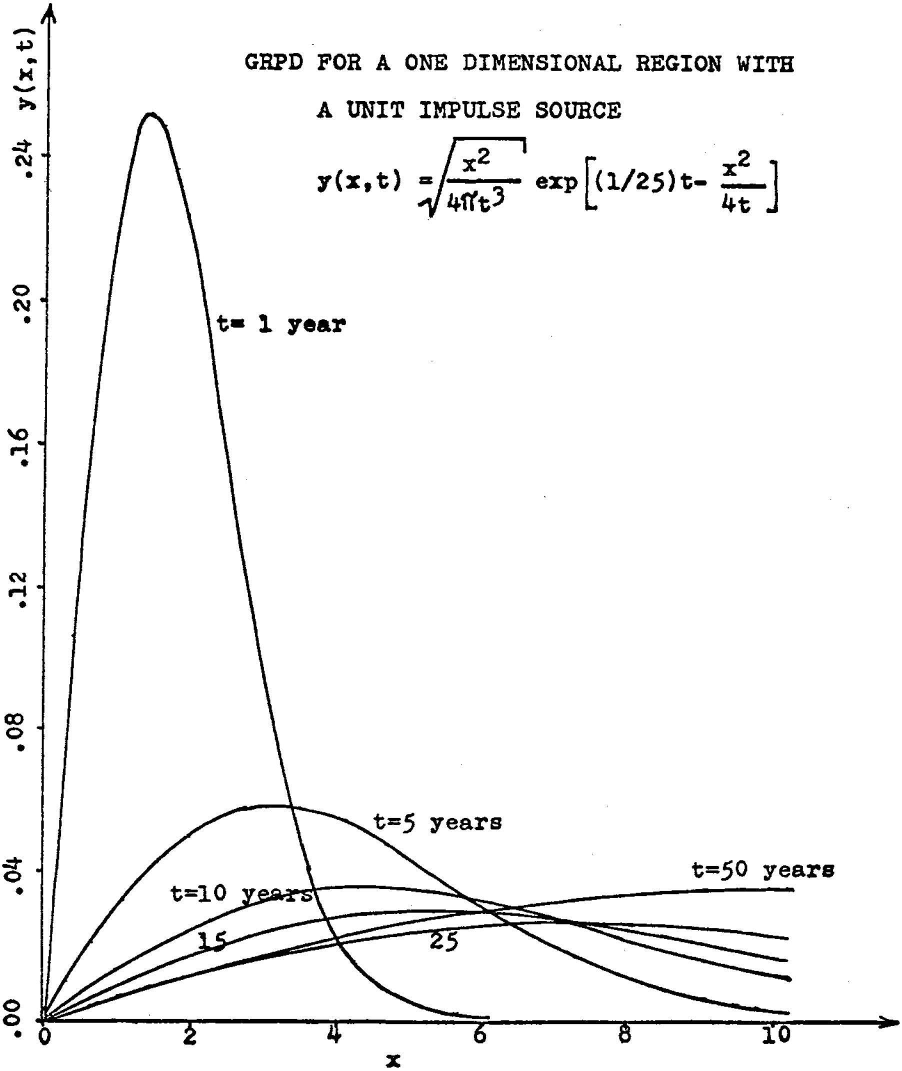

GRPD FOR A ONE DIMENSIONAL REGION WITH
A UNIT IMPULSE SOURCE
$y(x,t) = \sqrt{\frac{x^2}{4\pi t^3}} \exp\left[(1/25)t - \frac{x^2}{4t}\right]$
y(x,t)
.24
.20
.16
.12
.08
.04
.00
t= 1 year
t=5 years
t=10 years
15
25
t=50 years
0
2
4
6
8
10
x

FIGURE 4

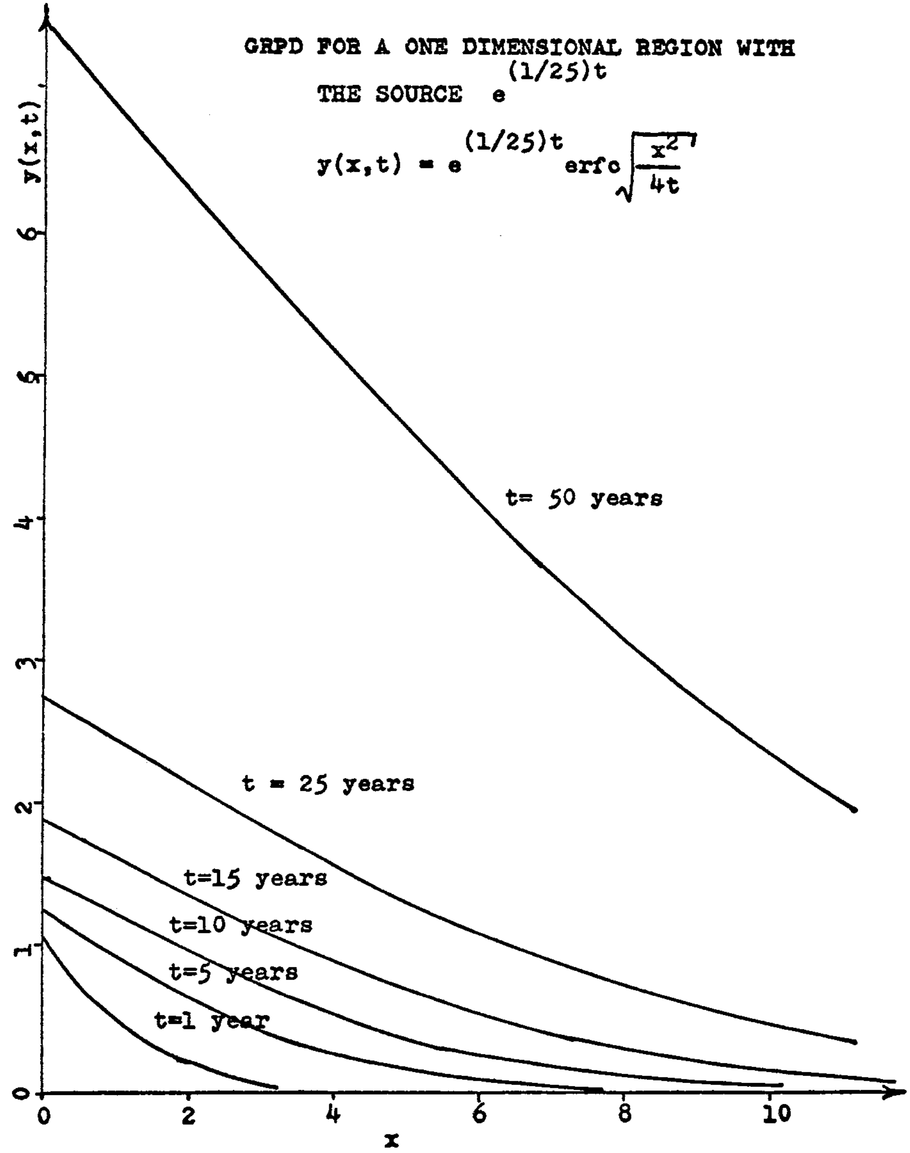

GRPD FOR A ONE DIMENSIONAL REGION WITH
THE SOURCE $e^{(1/25)t}$
$y(x,t) = e^{(1/25)t} \operatorname{erfc}\sqrt{\frac{x^2}{4t}}$
t= 50 years
t = 25 years
t=15 years
t=10 years
t=5 years
t=1 year
y(x,t)
x
0
1
2
3
4
5
6
0
2
4
6
8
10

FIGURE 5

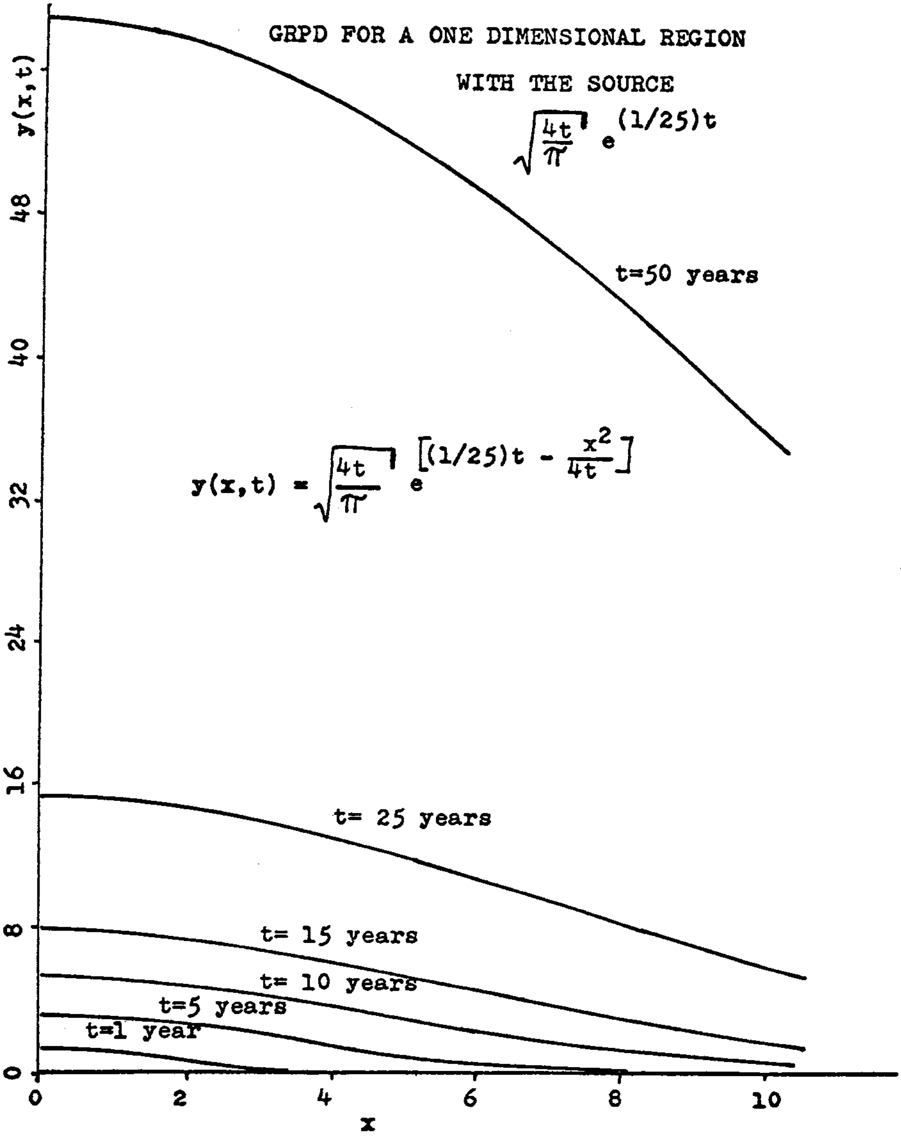

GRPD FOR A ONE DIMENSIONAL REGION
WITH THE SOURCE
$\sqrt{\frac{4t}{\pi}}\, e^{(1/25)t}$
t=50 years
$y(x,t) = \sqrt{\frac{4t}{\pi}}\, e^{\left[(1/25)t - \frac{x^2}{4t}\right]}$
t= 25 years
t= 15 years
t= 10 years
t=5 years
t=1 year
y(x,t)
0
8
16
24
32
40
48
0
2
4
6
8
10
x

FIGURE 6

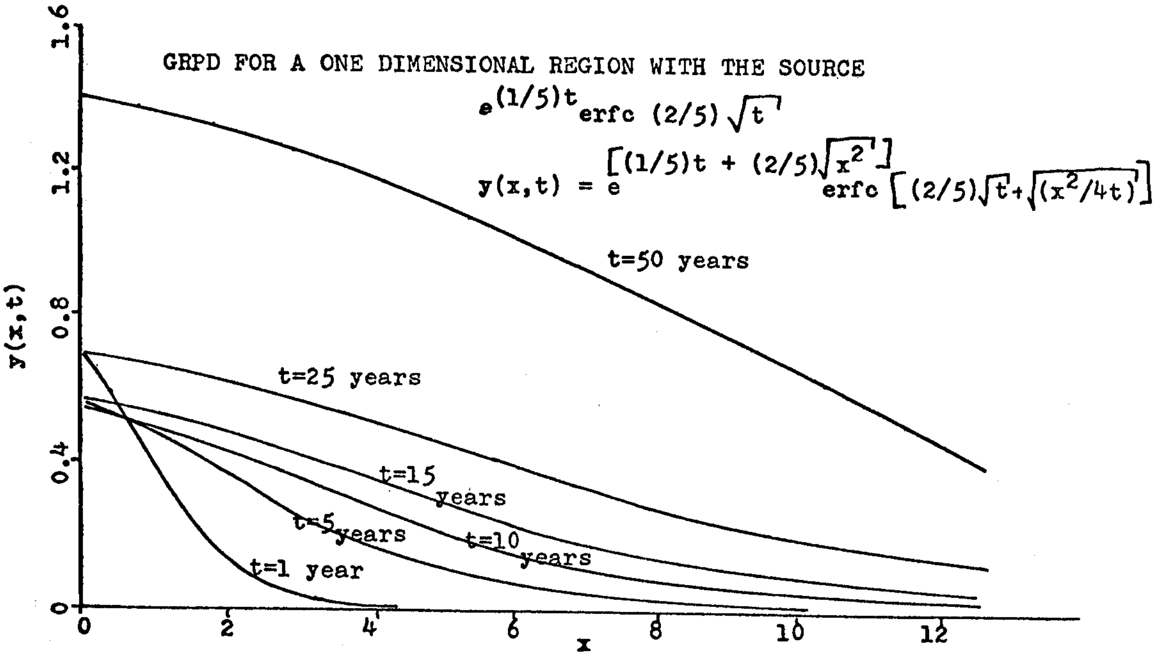

GRPD FOR A ONE DIMENSIONAL REGION WITH THE SOURCE
$e^{(1/5)t} \operatorname{erfc} (2/5)\sqrt{t}$
$y(x,t) = e^{\left[(1/5)t + (2/5)\sqrt{x^2}\right]} \operatorname{erfc}\left[(2/5)\sqrt{t} + \sqrt{(x^2/4t)}\right]$
t=50 years
t=25 years
t=15 years
t=10 years
t=5 years
t=1 year
y(x,t)
1.6
1.2
0.8
0.4
0
x
0
2
4
6
8
10
12

FIGURE 7

$$(29)\quad \begin{cases} D\ (1/r)(\partial/\partial r)[r\ \frac{\partial y(r,t)}{\partial r}] = \partial y(r,t)/\partial t - A\ y(r,t) \\ y(r,0) = 0 \\ y(0,t) = y_s(0,t) \\ y(r,t) \text{ bounded for } r \geq 0 \text{ and } t \geq 0 \end{cases}$$

If the source is an impulse function $y_s(0,t) = q_o\ \delta(t)$, then the solution is

$$(30)\quad y(r,t) = (q_o/2t)\ e^{At} e^{-r^2/4Dt}$$

Figure 8 illustrates this case for A = 0.04 or 4% per annum, T = 1 year, and the radial distance from the origin is in relative units r. After the impulse, the location of the source experiences a quick drop of GRPD. The bottom of the local business cycle occurs at about 30 years. Thereafter, the growth effect will overtake the slump. At r = 4.5 the GRPD stays about fixed from t=5 years to t=25 years. There is decay inside of this point and growth outside. It takes about 50 years before the inner area recovers to the level it had after 10 years of the initial impulse excition. This may in part caricature a process of center city decay and the growth of suburbia.

The Total Contribution to the Gross Regional Product by a Dynamically Specified Source

A central problem for economic planning for a region is estimation of the total GRP contribution of a source industry over time. By the above growth-diffusion model can be found both that portion of GRP generated directly by the source, and that portion induced by the source. The former component may be called the "direct benefit" and the latter component may be called the "induced benefit" or "indirect benefit." Let $Y_s(t)$ be the direct benefit and let Y(t) be the total GRP; then the induced benefit is $Y(t) - Y_s(t)$ and the percentage induced GRP would be $100\ [1 - Y_s(t)/Y(t)]$.

Consider the case illustrated in Figure 8 where the source is assumed to have a radius L. Its total dollar volume, then, is $Q_o = \pi L^2 q_o$. Integration of equation (30) from r=0 to r=∞ over the whole region yields the total GRP as

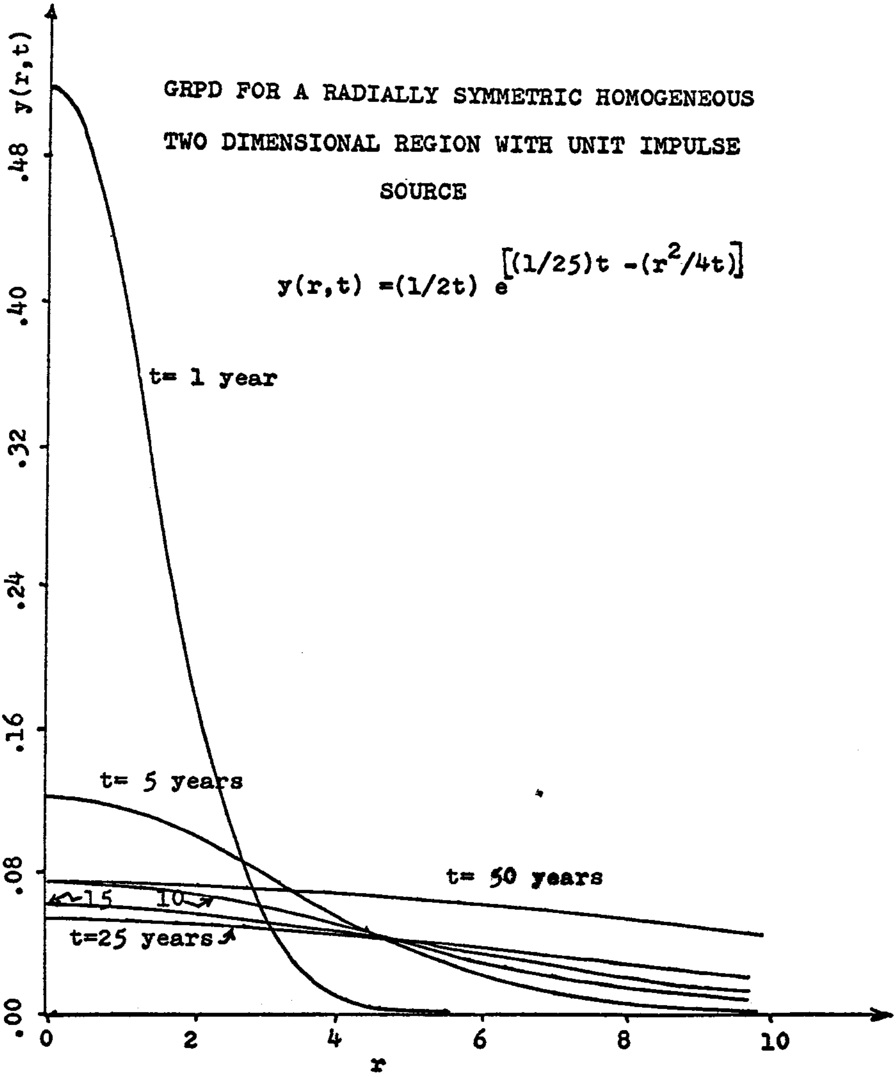
$$Y(t) = 2\pi\ q_o\ D\ e^{At}$$

Noting Q_o above and assuming $D = L^2/T$

$$Y(t) = (2Q_o/T)\ e^{At}$$

$$Y_s(t) = 2Q_o/T$$

Thus, the % induced GRP is $100\ [1 - \exp(-At)]$.



GRPD FOR A RADIALLY SYMMETRIC HOMOGENEOUS TWO DIMENSIONAL REGION WITH UNIT IMPULSE SOURCE
$y(r,t) = (1/2t)\, e^{[(1/25)t - (r^2/4t)]}$
t= 1 year
t= 5 years
t= 50 years
15
10
t=25 years
y(r,t)
.48
.40
.32
.24
.16
.08
.00
0
2
4
6
8
10
r

FIGURE 8

Applications of One Dimensional Growth-Diffusion Models to Two Dimensional Non-Isotropic, Non-Homogeneous Problems

In a two dimensional region, diffusion and growth of economic activity often occurs in linear pattern along roads, streets, shorelines, valleys, etc. In such cases the longitudinal diffusivity will be higher than the transversal diffusivity along the secondary paths crossing the main paths.

Figure 9.1 illustrates a map of an imaginary river basin in which the main artery is a large river, navigable by big ships, and the numerous small side branches are navigable only by small ships. In Figure 9.II the river basin is mapped into an orthogonal system with the main artery in the x-direction and the side branches in the z-direction. The harbor at the mouth of the main river is considered as the primary source of economic activity, and its diffused effects along the main artery generate secondary sources for the side branches of the river basin. The diffusivity D_x is assumed to be considerably higher than diffusivity D_z. Equation (26) can be generalized to this situation. Let the Laplace transform of the main source at the main harbor be $\mathcal{Y}_s(0,0,s)$. Along the main artery z=0 and

$$\mathcal{Y}(x,0,s) - \mathcal{Y}_s(0,0,s)\ \exp\left[-\sqrt{[(s-A_x)/D_x]\ x^2}\right]$$

Note the growth rate A_x along the artery can be different from that, A_z, for the side branches. For side branches, the source is the economic activity along the main artery. Thus,

$$\mathcal{Y}(x,z,s) = \mathcal{Y}(x,0,s)\ \exp\left[-\sqrt{[(s-A_z)/D_z]\ z^2}\right]$$

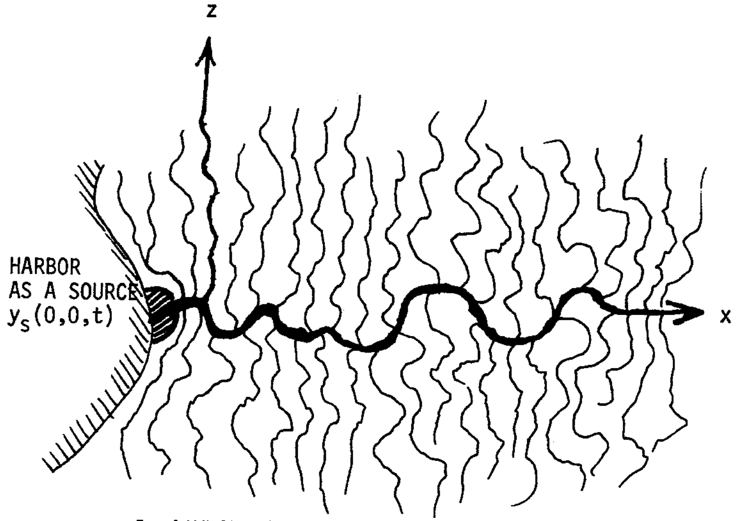
$$= \mathcal{Y}_s(0,0,s)\ \exp\left[-\sqrt{[(s-A_x)/D_x]\ x^2} - \sqrt{[(s-A_z)/D_z]\ z^2}\right]$$

y(x,z,t) is obtained by taking the inverse Laplace transform of the above expression.

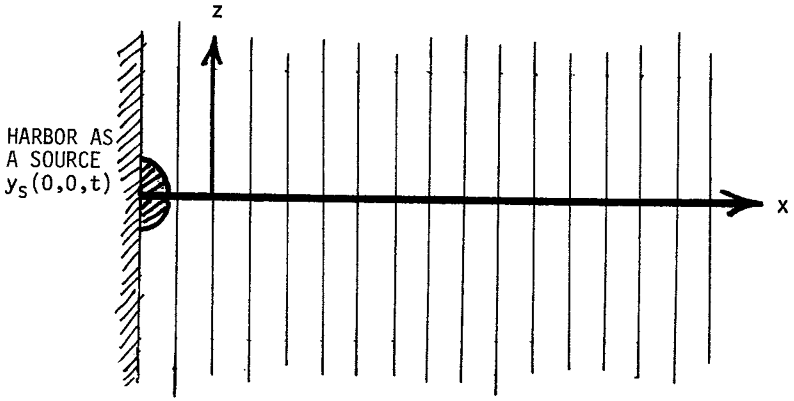
Further generalization of the above is illustrated in Figure 10 which considers a network of diffusion paths. The primary source is at x=0 for the x-branch. The diffusion process generates a secondary source at $x=x_1$ for the z-branch, and so on. Following the reasoning used in the previous case, one obtains for the impact location at $w=w_1$ the following expression for GRPD:

$$\mathcal{Y}(w_1,s) = \mathcal{Y}(x=0,s)\ \exp\left[-\sqrt{[(s-A_x)/D_x]\ x_1{}^2} - \sqrt{[(s-A_z)/D_z]\ z_1{}^2} - \sqrt{[(s-A_u)/D_u]\ u_1{}^2} - \sqrt{[(s-A_v)/D_v]\ v_1{}^2} - \sqrt{[(s-A_w)/D_w]\ w_1{}^2}\right]$$

There exists a great variety of applications with this basic approach to non-homogeneous and non-isotropic problems that could be approximated by one dimensional piecewise homogeneous networks. Non-homogeneous, non-isotropic problems can also be approximated by the mosaic approach in which each sub-region in the mosaic has an arbitrary shape with a connected interior with homogeneous and isotropic growth-diffusion characteristics.



I. ACTUAL MAP OF A RIVER BASIN



II. ORTHOGONAL MAPPING OF THE MAP OF A RIVER BASIN

FIGURE 9

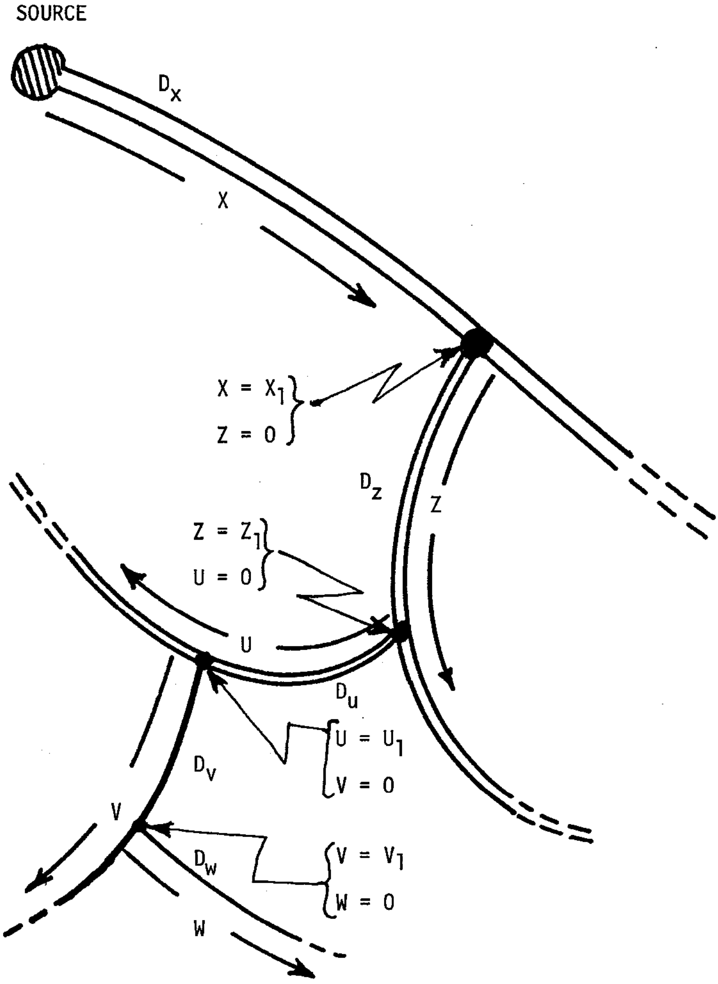


FIGURE 10

One can also introduce appropriate boundary properties between adjacent sub-regions in the mosaic.

Effects of Barriers Upon Diffusion and Growth of Regional Economic Activity

Barriers can change and restrict diffusion and growth of GRPD over a region and in the interfaces between regions. Geographical barriers such as mountains, oceans, rivers, and swamps and political barriers such as borderlines between nations, and trade restrictions, for example, may partially transmit, partially reflect, or absorb GRPD [12]. Trade barriers may have similar properties. Intraregional social and cultural groupings also generate various barrier effects. The growth-diffusion model and its modifications can naturally incorporate these various boundary effects in contrast to most conventional economic theories.

Figure 11 illustrates the effect of a reflective boundary between two regions. Each region has its own diffusivity. The river serves as a geographical as well as political barrier. A source $y_{1s}(0,t)$ at the shore of the river in the region 1 generates its GRPD given by $y(r_1,t)$. The source is partially transmitted over a bridge to the region 2 setting up there a secondary source $y_{2s}(0,t)$. This, in turn, generates $y(r_2,t)$.

Final Remarks

It is clear that one could introduce several other points of discussion about the model and its extensions. This will not be attempted here. One point may be emphasized as a conclusion. The linear growth-diffusion model introduced here is not a necessary but is a sufficient starting point for introducing a concept of diffusion into regional economic theory. As such, it may serve as a base for extensions in several different directions, and provide the possibility of combining several economic, socio-political, geographic, and cultural aspects of human ecology into a relevant picture of a regional economic development.

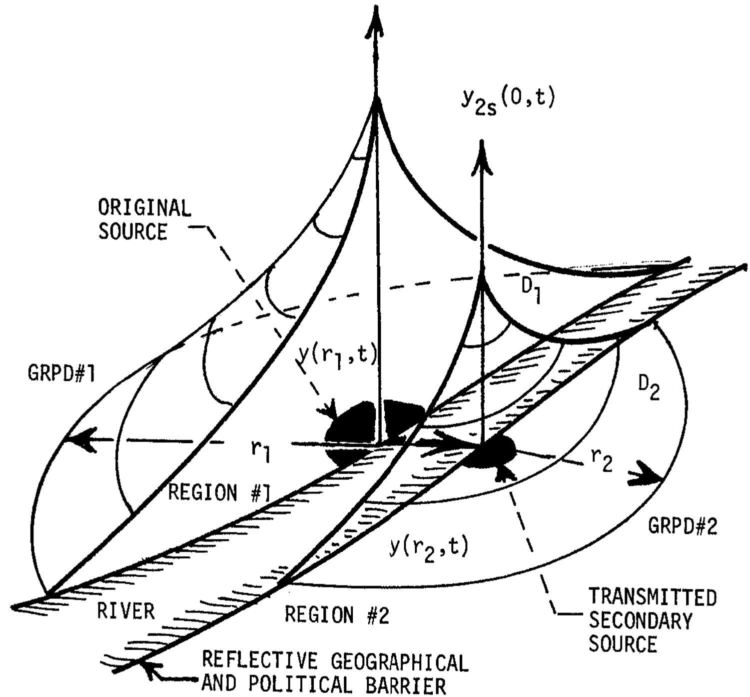

y1s(0,t)
y2s(0,t)
ORIGINAL SOURCE
D1
y(r1,t)
GRPD#1
D2
r1
r2
REGION #1
GRPD#2
y(r2,t)
RIVER
REGION #2
TRANSMITTED SECONDARY SOURCE
REFLECTIVE GEOGRAPHICAL AND POLITICAL BARRIER

FIGURE 11

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