THE CASE HISTORY OF A MONTE CARLO SIMULATION: INTERPRETATION AND CALIBRATION OF THE MODEL OF THE IMPERFECT CENTRAL PLACE PLANE*

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The model of the imperfect central place plane is a stochastic location process that generates point patterns characterized by regularity in the spacing between neighboring points. Dacey [1] briefly describes the model, considers its application to the urban pattern formed by largest places in Iowa and provides sufficient empirical analysis to establish that the model merits detailed investigation. This paper provides further details on two aspects of the model.

The first part of this paper more clearly places the model in its historical and geographical perspectives. On one hand, it is shown that the model takes account of fundamental characteristics of the historical development of urban patterns. On the other hand, relations are established between this model and the more general principles of central place theory. The relation of the model to theory has a critical role in the interpretation of elements of the model.

The latter part of this paper considers problems reflecting that the model identifies a sufficiently complicated stochastic process that tractable expressions have not been obtained for properties of random varibles describing the theoretical point patterns. While approximating expressions might be obtained for at least some of the properties, this approach has not been fully investigated. Instead, the location process is studied by the use of simulation procedures to generate synthetic patterns, and measurements from these synthetic patterns provide sample estimates of properties of the location process. This paper considers the design of an appropriate simulation procedure.

One reason for the detailed examination of the simulation design is that the search for a literature that could guide the construction and use of the synthetic pattern generator was not fruitful. While a possible explanation is that simulation of the model of the imperfect central place plane presents problems that typically do not arise in other simulations of pattern, an examination of the geographic uses of simulation failed to sustain the hypothesis. Instead, the interpretation and calibration of pattern simulators appears to involve a common body of problems that include the manner of (1)

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specifying a simulation space, (2) handling boundary effects and (3) scaling the model to known units. While the way these considerations arise in various simulations may vary in degree and context, the underlying methodological issues seem inescapable consequences of the relatively undeveloped status of most geographic theories. The nature of several of these fundamental problems is clarified by the detailed investigation of procedures involved in the simulation of a specific model.

Historical Perspective

The model of the imperfect central place plane is intended to describe the urban pattern formed by a collection of largest cities and towns in a region. The model is particularly suited to regions located in the eastern portion of the trans-Mississippi area where the conditions of classical central place theory are approximately satisfied. However, the model modifies central place principles in order to take account of fundamental characteristics of the historical development of the urban patterns in midwestern states. Four of these characteristics are mentioned briefly.

One characteristic is that these states were surveyed prior to settlement and, in particular, the boundaries of each county were designated before it was opened to settlement. Moreover, county boundaries have been highly stable. A second characteristic is that the seat of each county government was designated soon after the county was opened to settlement and, commonly, a site was selected near the geographic center of the county. A third characteristic is the early growth of county seat places. The presence of administrative and judicial functions in a county seat place provided an initial agglomeration of activities, which was an impetus to early growth. Although there are conspicuous exceptions, the general tendency was for the county seat function to remain at the initially selected site, which was reinforced by early growth of county seat places. A fourth characteristic is the continued advantage of early growth so that in contemporary urban patterns the largest place in a county is typically the county seat place. The important consequences to the model building are that the largest places tend to be county seat places and these places tend to be located near the centers of counties.

Modeling of County Structure

The model of the imperfect central place plane is the formulation of a location process that recognizes the distinction between county seat places and other, non-county seat places and utilizes county structure as an element of the geographic environment that affects the location of county seat places.

Central place principles are incorporated into the model by interpreting county structure as a cell structure composed of polygons that are identical in size and shape. When the cells are regular hexagons or, possibly, squares, county structure corresponds to the structure of market areas in a central place system. While use of a cell structure for a system of counties is an abstraction from geographic reality, the abstraction bears a relation to the geographic situation wherever the partitioning of a state into counties resembles a tesselation of regular cells. Although county size varies from state to state, within each of the trans-Mississippi states the shape and arrangement of counties approximates a tesselation of regular polygons. While it is possible to design a test that partially evaluates the adequacy of this abstraction, a more critical test is the degree to which the theoretical pattern generated by the model corresponds to observed urban patterns.

The model that is developed in this paper replaces county structure by regular hexagons or by squares. The type of interdependence that is assumed to hold between county structure and central place principles is exemplified by the answer given to the following question. If the division of midwestern states into counties produced a pattern having no resemblance to a regular lattice structure, would central place principles underlie the urban location process? The model of the imperfect central place plane is predicated upon the universal validity of central place principles.

The Model

The model of the imperfect central place plane takes into account the county structure of a region and the distinction between county seat and non-county seat places. The basic elements of the model are cell, CS-place and O-place and in the interpretation and subsequent application of the model these elements are interpreted, respectively, as county, county seat place and non-county seat (or other) place. These elements are defined in terms of a point lattice L embedded in the euclidean plane E and a collection P of places that are treated as points located on the plane. Each place is a CS-place or an O-place and the locations of these places are realizations of a stochastic process which is identified with respect to points of the lattice.

A cartesian coordinate system is assumed, and the lattice is defined by translation periods τ_1 and τ_2 whose orientations differ by the angle γ. The collection of lattice points is $L = \{\ell(u, v) \mid u, v \text{ integers}\}$ where, in vector notation

$$\ell(u, v) = u\vec{\tau}_1 + v\vec{\tau}_2.$$

and, in cartesian coordinates, $\ell(u, v) = (u\tau_1 - v\tau_2 \cos\gamma, v\tau_2 \sin\gamma)$. The lattice point $\ell(u, v)$ is called the "geographic center" of the cell $c(u, v)$ that consists of all points of the plane closer to the lattice point $\ell(u, v)$ than to any other lattice point. Hence, the cells form Dirichlet regions of the plane, and put $C = \{c(u, v) \mid u, v \text{ integers}\}$. Let η represent the area of each cell, and η is determined by the lattice parameters τ_1, τ_2 and γ.

The entities of the model of the imperfect central place plane are the collection P of places that are divided into CS-places and O-places and the collection C of cells that are defined with respect to a point lattice L. The lattice has infinite extent so that it is not meaningful to enumerate the numbers of cells, CS-places and O-places. The relative numbers of such places may, however, be quantified by use of a measure such as the average number of places per cell. Let ρ denote the average number of CS-places per cell and let μ denote the average number of O-places per cell. It is specified that $0 \leq \rho \leq 1$ and $\mu \geq 0$, but the reasons for the variation of ρ between 0 and 1 may require clarification. When the application of the model is to the pattern formed by the m largest places in a region having n counties, it is not necessary that these m places include all n county seat places. This is clearly the case if $m < n$. Moreover, since the n largest places in an n-county region will frequently include non-county seat places, $m > n$ does not necessarily imply that the set of m largest places in a region includes all county seat places.

Three assumptions relate the location of places in P to the cell structure C.

A-1. The locations of O-places form a Poisson point process on the plane with density μ.

A-2. Each cell generates one CS-place with fixed probability ρ and generates no CS-place with probability $1 - \rho$.

A-3. Given that the cell c (u, v) generates a CS-place, the location of this place in cartesian coordinates is $(u \tau_1 - v \tau_2 \cos \gamma + X_1, v \tau_2 \sin \gamma + X_2)$, where X_1 and X_2 are random variables defined by

$$X_1 = X \cos \theta, \quad X_2 = X \sin \theta .$$

where θ is uniformly distributed on $(0, 2\pi)$ and X has the half-normal distribution with

$$P\{X \leq t\} = (2/\pi\sigma^2)^{\frac{1}{2}} \int_0^t e^{-x^2/2\sigma^2} dx, \quad t \geq 0, \sigma > 0.$$

Notice that the locations of places are treated as independent events and that the CS-place associated with origin $\ell(u, v)$ is not necessarily located in c (u, v).

The three kinds of parameters of the model of the imperfect central place plane are

(a) the parameters τ_1, τ_2 and γ defining the regular point lattice L,

(b) the density parameters ρ and μ, and

(c) the parameter σ associated with the random variable X.

The pattern of places generated by this model is described as random variables that represent spacing measures between entities located within a convex region β of the imperfect central place plane, and β may or may not coincide with the euclidean plane. The random variables are of two different kinds.

One kind represents distances from sample points located in β to neighboring places located in β, where the locations of these sample points are realizations of a Poisson point process on the (imperfect central place) plane. Let ω represent a sample point located in β and three random variables are

U_i represents distance from ω to the i^{th} nearest CS-place in β,

V_i represents distance from ω to the i^{th} nearest O-place in β, and

T_i represents distance from ω to the i^{th} nearest place of either type in β.

The other kind of random variables correspond to order distances from place to neighboring places, and these random variables are

U_i^* represents distance from a CS-place in β to the i^{th} nearest (other) CS-place in β,

V_i^* represents distance from an O-place in β to the i^{th} nearest (other) O-place in β, and

T_i^* represents distance from a place in β to the i^{th} nearest (other) place of either type in β.

Each of the six order distances is defined for $i = 1, 2, \ldots, \kappa$ so that the pattern of places in a region β is described by the probability distributions and moment properties of κ orders of these six random variables.

Description of the Theoretical Pattern

The basic hypothesis of the model building is that there exists a region β of the imperfect central place plane such that spacing attributes of an observed pattern of m largest places correspond to, or may be approximated by, a realization of the model of the imperfect central place plane in the region β. However, several critical problems arise in the specification of β.

First, suppose β is specified as the euclidean plane. For this region properties of V_i and V_i^* are known, but I am not able to derive useful properties of the other random variables. While it might be possible to generalize results given in Dacey [2] to obtain integral expressions for U_i and U_i^*, these expressions would undoubtedly be intractable and of little use in numerical analysis. While approximating expressions might be obtained for some properties of these random variables, this approach has not been fully investigated because the use of a simulation procedure appears more productive.

A simulation procedure involves the construction of synthetic point patterns having properties that approximate those of the theoretical pattern of places on the imperfect central place plane. Properties of the six random variables describing the theoretical pattern are estimated from measurements taken from the synthetic patterns. Synthetic patterns are necessarily generated on a bounded region.

If the size and shape of the region β is specified, the model is such that it is easy to design a simulation procedure that generates acceptable synthetic point patterns. The difficult aspect is the specification of a simulation space β that contains a synthetic pattern having properties that are comparable with properties obtained from an observed urban pattern. Most of the difficulties are attributable to the effects of boundaries on distance measures from both synthetic and observed urban patterns.

Before considering specific problems, it may be useful to comment first on the general problem of edge effects. The patterns involved in geographic research are composed of a collection of objects (which are commonly abstracted to point, line or area symbols) located on a portion of the earth's surface (which is commonly represented by a two dimensional map area that contains the symbols corresponding to the locations of objects). The description and analysis of point patterns uses procedures that take account of the arrangement of objects--the location of objects with respect to each other--and the dispersion of objects--the location of objects with respect to the map area containing the objects. Suppose the arrangement of objects is expressed as a function of the distance between pairs of objects (as the random variables U_i^* and V_i^*) and dispersion of objects is expressed as a function of the distances between uniformly located sample points and objects (as the random variables U_i and V_i). Then it may be shown that arrangement and dispersion can be varied independently and independent of the size and shape of the map region. However, it is unlikely that the size or shape of the map region can be changed without also changing the dispersion of a pattern. When a change in the map area alters dispersion, pattern description is intimately dependent upon the specification of the region containing the pattern. As a consequence, the comparison of synthetics and empirical patterns requires calibration of the model so that comparable measurements are obtained from the two pattern types.

One approach to edge effects is to construct the simulation space β so that it has exactly the same dimensions as the region containing the observed pattern that is being compared with the model of the imperfect central place plane. Such a simulation space cannot in general be constructed, but to verify this conclusion it is first necessary to identify properties of the observed patterns that will be compared with the model.

Selection of the Study Region

Interpretation and analysis of the model imposes constraint upon the specification of patterns that are comparable with the theoretical pattern of places in the imperfect central place plane. First, observed data are not obtained from an urban pattern but from its map representation. Because county structure varies from state to state it is assumed that the map region roughly corresponds to the map area of a state in the trans-Mississippi area where the underlying conditions of central place theory are approximated.

The map is at a scale that depicts the locations of towns and most cities by point symbols. Because the model does not take urban agglomerations into account, it is necessary to combine suburbs and other spatially contiguous cities so that the universe of places consists only of areally distinct cities and towns. The study region is defined with respect to an urban system defined by the m_0 largest, areally distinct places in a state, though other criteria may be used to delimit an urban system.

The map region is a large rectangle containing the centers of n counties such that the area of the rectangle equals the area of these n counties. Though it may require some shifting of boundaries to satisfy this criterion, the criterion on area is critical because it will be used in calibrating the model. The urban pattern is defined by the map representation of the point locations of the largest places within this map region. Let B denote this region and m denote the number of places in B.

Interpretation of the Model

The interpretation of elements of the model utilizes the well-developed structure of central place theory and the assumed relation between county structure of states and the cell structure of central place theory. The model is interpreted for the collection of m largest places in a map region B. This region covers part or all of a state having the following characteristics: it is partitioned into non-overlapping counties, state law requires that each county has exactly one urban place that is the seat of county government, and the county seat place is located within the county it governs. When applied to a map pattern formed by the urban places in a region B, the elements of the model are given the following interpretations:

(a)
- lattice cell - county
- CS-place - county seat place
- 0-place - non-county seat place
- lattice point - geographic center of county
- $Y = (X_1^2 + X_2^2)^{\frac{1}{2}}$ - distance from geographic center of county to county seat place

The pertinent parameters of the collection of m largest places in a map region B are

(b)
- n - number of counties in the region
- x - number of county seat places
- y - number of non-county seat places
- a - area of region

The pattern formed by the m largest places is described by sample estimates of the six random variables describing the theoretical pattern of places on the imperfect central place plane. Sample estimates of U_i, V_i and T_i are obtained from observed distances from sample points that are uniformly and independently located on the map region B to, respectively, i^{th} nearest county seat place, i^{th} nearest non-county seat place and i^{th} nearest place of any type. Estimates of U_i^*, V_i^* and T_i^* are obtained from observed i^{th} order distances between pairs of county seat places, non-county seat places and places of any type. The pattern of m largest places in a map region B is described by κ orders of distances for each of the six types of spacing measures. Though these spacing measures may be expressed in an arbitrary metric, it is convenient to select the metric such that $a = n$.

Specification of Parameters

The six parameters of the model are τ_1, τ_2, γ, ρ, μ and σ. In addition there is need to set the number κ of order distances. Combining the interpretation of the model (a) with the values (b) for parameters of an urban pattern suggest that ρ is estimated by x/n and, when $a = n$, μ is estimated by y/n. The interpretation (a) does not, by itself or in conjunction with the parameter values (b), yield properties of the lattice structure. The three lattice parameters are specified by placing the interpretation (a) in a central place context. For a central place structure, L is a hexagonal or, possibly, a square point lattice. For these lattices, $\tau_1 = \tau_2 = \tau$, say, and γ has the value $2\pi/3$ for the hexagonal and $\pi/4$ for the square lattice.

To obtain a numerical value for the parameter σ, the ideal circumstance would use a theory that relates the dispersion of county seat places around the geographic centers of counties. Lacking this theory, the parameter σ of the half-normal distribution is estimated from empirical data that give the distance from the center of each county in the map region B to its county seat place. The estimate of σ is expressed in the same unit of measurement as the statistics describing the map pattern. This unit sets the metric for the model.

The postulated relation between the cell structure of the central place lattice and the county structure of a region suggests that the lattice L is constructed so that the area η of each primitive lattice cell is estimated by a/n, the mean area of counties. Given this value and the type (hexagonal or square) of lattice, the translation period τ is obtained by simple algebra. The unit of measurement for τ is the same as used for σ.

The remaining problems are to identify a simulation space β of the central place plane that is described by random variables having properties that are comparable with the statistics describing the urban pattern in the region B and to set a value for κ.

Specification of a Simulation Space

It is given that the map region B is a rectangle with sides of lengths a_1 and a_2 and area $a = a_1 a_2$ that contains the centers of n counties having total area a. The preceding interpretations of the model seem to suggest that the simulation space β is constructed as a rectangle with sides of lengths a_1 and a_2 that contains n lattice points of L. However, in general there does not exist on the imperfect central place plane a region having these properties--for example, when L is a square point lattice and n is odd. This verifies the earlier assertion that calibration of the model cannot be handled by the simple expedient of constructing a region β that has exactly the same dimensions as the map region B.

Before constructing the regions β and B the obstacles that confront specification of these regions are summarized. Because of inability to derive properties for random variables, the model of the imperfect central place plane is studied by simulation of the theoretical pattern. This synthetic pattern necessarily occupies a bounded region. Boundary effects cannot be ignored because estimates of properties of the location model are based on measurements from synthetic patterns and these estimates are affected by the shape and size of the simulation space. While this would not bias the comparison of theoretical and observed patterns when the simulation space has the same areal dimensions as the map region, a simulation space that has properties of the map region cannot in general be constructed. Hence, the present need is to construct a simulation space and a map region that control or minimize the bias attributable to boundary effects.

This type of calibration problem probably arises in nearly all geographic simulations of pattern. While I cannot identify a literature that comments on procedures for controlling boundary effects, the problem should not be ignored. If boundary effects are not taken into account, it may not be possible to identify properties of the model being simulated. Calibration is imperative because it is not clear what conclusions can be obtained from uncalibrated models.

The procedures used to calibrate the present model and to control boundary effects are not universally applicable. They do, however, provide one basic strategy for controlling boundary effects when the simulation space and the map region are defined as rectangles. While there are several other reasons for using a rectangular shaped region, the primary one is that a relatively simple transformation, frequently used in theoretical physics, removes boundary effects. Opposite boundaries of this region are joined to form a helical torus (anchor ring). The resulting surface has no edges and hence, no edge effects. Though other types of biases accompany the use of the torus, they are easier to control.

When a region is mapped onto a torus, distance between points in the region is defined as distance on the surface of the torus. To define this distance, consider a rectangle b with vertices at (0, 0), (0, y), (x, y) and (x, 0). If $p_1 = (x_i, y_i)$ and $p_2 = (x_2, y_2)$ are points of b, the toroidal distance

$d(p_1,p_2)$ between p_1 and p_2 (i.e., the distance on the surface of the helical torus obtained by joining opposite edges of b) may be obtained in the following way. Let x_m denote the minimum of $(x_1 - x_2)^2$, $(x_1 - x_2 - x)^2$ and $(x_1 - x_2 + x)^2$, and let y_m denote the minimum of $(y_1 - y_2)^2$, $(y_1 - y_2 - y)^2$ and $(y_1 - y_2 + y)^2$. Then $d(p_1,p_2) = (x_m + y_m)^{\frac{1}{2}}$.

The map region B is a rectangle. For the purpose of comparing the pattern of places in the map region with the theoretical patterns defined by the model of the imperfect central place plane, the map region is mapped onto a helical torus so that spacing measures taken from the region B are toroidal distances. The next task is to construct a simulation space β that yields measurements that are comparable with the toroidal spacing measure describing the pattern of places in the map region B.

Construction of the Simulation Space

The locations of places on the imperfect central place plane are described by a two dimensional probability density surface. When L is a regular point lattice this surface may be generated by rotation and translation of the probability density surface on any unit cell, Figure 1, of the lattice. So that the simulation space β retains this property of the model, β is constructed as a rectangle with sides parallel to rows and columns of lattice points in L and lengths of sides satisfying the property that the number of lattice points of L that are in β is not changed by any translation of B. Figure 1 illustrates typical constructions of β for hexagonal and square point lattices; the region β for the hexagonal lattice must contain even numbers of rows and columns of lattice points.

Opposite edges of the region β are joined to form a helical torus. Simulations of the model generate synthetic patterns on the surface of this torus. Properties of the six random variables describing the model are estimated from measurements on these synthetic patterns and these measurements are ordered, toroidal distances. Properties of the torus are used to set the lengths of sides of β and the number κ of order distances used to describe the theoretical pattern of places.

The size of β is considered first. The specification of size makes use of the toroidal property that any point x on the surface of a torus can, by judicious cutting of the torus, be a point located at the center of a rectangle. Because of this property, if β has sides of lengths $2A_1$ and $A_2 \geq 2A_1$ and if all random variables describing the theoretical pattern of places on the imperfect central place plane are less than A_1 with probability 1, then it may be shown that these random variables are identical in probability with the corresponding random variables defined on the toroidal mapping of the space β. In this case, the use of a torus does not produce bias estimates of properties of the random variables. However, none of the six random variables are less than A_1 with probability 1 for any finite A_1 so that the toroidal construction yields bias estimates of properties of each of the six random variables.

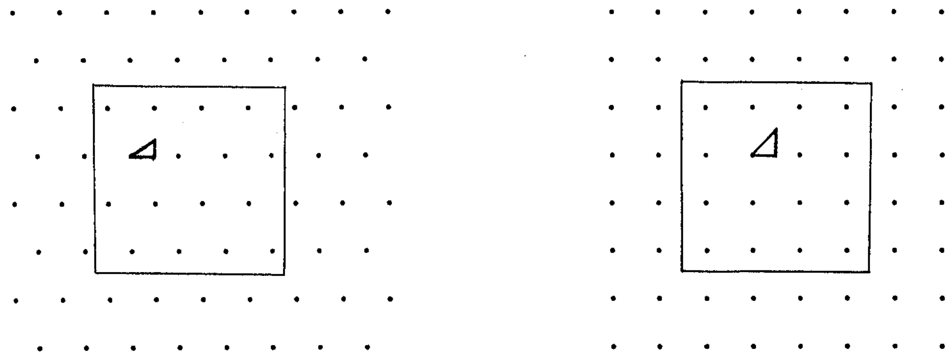

Figure 1. Examples of the construction of region B for hexagonal and square lattices. Typical unit cells are also illustrated.

Though bias is always present, it is possible to construct a space β that is "sufficiently large" that the degree of bias is within a prescribed limit. To see this, let X_i represent the i^{th} order of one of the six random variables. For fixed positive $\varepsilon_\beta < 1$ there is a sufficiently large region β such that X_i is less than A_1 with probability $1 - \varepsilon_\beta$. The quantity ε_β is a measure of the effect that the size of a specified region β exerts on estimates of properties of the random variables. By putting $\varepsilon_\beta = .01$, $.001$ or some other small value, a region β can be constructed for which the toroidal mapping introduces only a negligible bias on estimates of properties of each of the six random variables.

Suppose β has sides of lengths $2A_1$ and $A_2 \geq 2A_1$. Given the allowable error ε_β, the following procedure defines the dimensions of a region β that is "sufficiently large" for K orders of distance measures.

The estimation of an A_1 that is sufficiently large for estimating properties of U_i makes use of the fact that $P\{U_K \leq A_1\} \geq 1 - \varepsilon_\beta$ if, and only if, any circle on the imperfect central place plane with radius A_1 contains at least K CS-places with probability greater than $1 - \varepsilon_\beta$. This probability is approximated by considering a circle centered on a lattice point having radius $R + 3\sigma$. Since a CS-point is associated with each lattice point with probability ρ, the circle with radius $R + 3\sigma$ contains exactly k CS-points with approximate probability

$$p_k(R) = \binom{L(R)}{k} \rho^k (1 - \rho)^{L(R)-k}$$

So, the probability that at least K CS-places are within distance R of a lattice point is approximately

$$P_K(R) = 1 - \sum_{k=0}^{K-1} p_k(R)$$

If the rectangular region β has sides with lengths $2A_1$ and $A_2 \geq A_1$, then $U_K \leq A_1$ with approximate probability $P_K(A_1)$. If A_1 is specified so that $P_K(A_1) \geq 1 - \varepsilon_\beta$ then β is considered sufficiently large for K orders of distances from sample points to CS-places with negligible bias attributable to the toroidal mapping. Also, the same approximation is usable for the random variable U_i^*.

An exact probability is obtained for the sample point to 0-place distances represented by the random variables V_i and V_i^*. On the central place plane the distance from a sample point or from an 0-place to the k^{th} nearest 0-place is less than R with probability

$$q_k(R) = [2(\pi\lambda)^k/(k-1)!] \int_0^R x^{2k-1} e^{-\pi\lambda x^2}\, dx$$

$$= \frac{(\pi\lambda)^k}{(k-1)!} \gamma(k, \pi\lambda R^2)$$

where $\gamma = \mu/\eta$ is the density of 0-places per unit area and $\gamma(k, z)$ is the incomplete gamma function. If A_1 is set so that $q_K(A_1) \geq 1 - \varepsilon_\beta$, then β is considered sufficiently large for K orders of distances from sample points to O-places and from O-places to O-places with negligible bias attributable to the toroidal mapping.

Clearly, if a region is sufficiently large for K orders of measurements to CS-places and to O-places, then it is sufficiently large for K orders of measurements to a place of any type.

A region β having sides with lengths at least as large as $2A_1$ is said to be sufficiently large for K orders of measurement when both $P_K(A_1) \geq 1 - \varepsilon_\beta$ and $q_K(A_1) \geq 1 - \varepsilon_\beta$.

Numbers of Order Distances

The evaluation of the model of the imperfect central place plane involves comparison of properties of κ orders of six random variables with similar properties of the corresponding statistics describing the observed urban pattern in B. Since the simulation space β may be made arbitrarily large, properties of the random variables may be obtained for any order of spacing measure. In contrast, the size of the map region B has already been set and used in calibrating the model. The maximum number of orders of toroidal distance measures that may be obtained from this region with negligible bias attributable to the toroidal construction is limited by its size. This number may be set by accepting the validity of the model of the imperfect central place plane and then using the probabilities P_K and q_K to establish the number of orders of statistics that may be estimated from region B with sides of lengths $2a_1$ and $a_2 \geq 2a_1$. The k^{th} order statistics are obtained from B only if $P_k(a_1) \geq 1 - \varepsilon_\beta$ and $q_k(a_1) \geq 1 - \varepsilon_\beta$.

If these two inequalities are not satisfied for k = 1, then it is necessary either to enlarge the region to which the model is applied, which necessitates a thorough recalibration of the model, or to increase the value of ε_β, which increases the allowable bias resulting from the use of a torus.

If this problem does not arise or suitable adjustments are made, suppose the two inequalities hold only for $K_B \geq 1$ orders of measurement. Then, a possible calibration of the model is to put $\kappa = K_B$ and to construct the simulation space β as the smallest rectangle that is sufficiently large for κ orders of distance measures. This completes the calibration of the model of the imperfect central place plane.

Sampling Procedures

Calibration of the model of the imperfect central place plane does not complete specification of the simulation procedure. Properties of the six random variables are estimated from sample measurements on the synthetic patterns generated by simulations of the model, and the simulation design needs to include a description of the sampling procedure. An appropriate sample design reflects the manner of using simulations to obtain descriptive properties of the model.

Each simulation of the model generates a single synthetic pattern of CS-places and O-places on the torus β. The order distance properties of this simulated pattern are obtained from measurements between, say, N_1 pairs of these places and between N_2 pairs of sample points and places. This procedure is repeated for M simulations of the model. Estimates of properties of each of the random variables describing the model are obtained from MN, $N = N_1 + N_2$, measurements of each order. The efficiency of a sample design undoubtedly depends on the number of simulations M, the number N of measurements on each simulated pattern and the size of the simulation space β. The sizes of M, N and β need to be set so as to obtain an efficient description of the theoretical pattern of places on the imperfect central place plane.

Other sampling problems arise in the empirical analysis of the model. These include specifying the number of sample observations on the urban pattern that are to be used to estimate each order statistic and the minimum number of orders to be used in empirical analysis of the model.

The design of sampling procedures, as well as the pertinent hypotheses to be evaluated by empirical analysis, are left for another paper.

REFERENCES

1. Dacey, M. F. "Regularity in Spatial Distributions: A Stochastic Model of the Imperfect Central Place Plane," in G. P. Patil, E. C. Pielou, and W. E. Waters (eds.) Proceedings of the International Symposium on Statistical Ecology, Volume 1. University Park: Pennsylvania State University Press, forthcoming.

2. Dacey, M. F. "A Probability Model for Central Place Locations," Annals of the Association of American Geographers, 56 (1966), 549-68.