THE OPTIMAL SIZE OF URBAN CENTERS: A THEORETICAL FRAMEWORK

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Assume a homogeneous plane in which an unspecified number of identical urban centers exists. The households inhabiting the centers are also identical, and their total number is given. The households are "rational" and choose a combination of consumers goods and residential attributes that maximize the satisfaction of their preferences subject to the condition that their income is not exceeded. The income of the households is assumed to be a function at first increasing, and then decreasing of the size of their city of residence. A state of spatial equilibrium exists, in the sense that no household can improve the level of satisfaction of its preferences (its optimal utility level) by moving to some other location in its city of residence or in some other city. A spatial equilibrium is assumed to exist when the optimal utility level attainable by each household is the same throughout the system [1, 2]. Call "optimal" the distribution of households among centers that maximizes the optimal utility levels of the households in the system.

This paper addresses itself to the problem of determining how many cities of which size are to exist for the distribution of households among them to be an optimal one. An optimal population distribution implies that the number of cities cannot be increased or decreased without lessening the optimal utility level of the households, or, conversely, that the cities in the system are of optimal size and their population cannot change without a decline in the optimal utility levels. This paper is therefore equally concerned with the determination of the optimal size of urban centers.

The Model

Let T_H indicate the total number of urban households in the system. Each household is made of m persons. Hence the total population in the system T_P is $T_P = mT_H$. Indicate by N the total number of cities in the system and by T_{PC} and T_{HC} the population size and the number of households in a city, respectively. If the cities are of identical size $T_{PC} = T_P/N = mT_H/N$. The object of this paper is to determine the optimal T_{PC} or, the optimal N, which is the same thing.

Assume that for all the cities in the system the places of work are

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concentrated in a central location. Let

$$u(z,q,s) = z^a q^b e^{-cs}; \quad 0 < a,b, < 1$$

be an indifference map specifying for a household the combinations of a composite consumption good z, of quantity of residential land q, and of distance s from the central location of its city of residence yielding to the household the same aggregate utility.

Assume that each household chooses the mix of z and q that maximizes u and satisfies a budget constraint $pz + r(s)q + ks = y$, where p is the price of the composite good, r(s) is the land value at distance s from the CBD, k is the unit transportation cost, and y is the income of the household.

By usual constrained optimation procedures the following optimal values $\bar{q}$ and $\bar{z}$ can be found:

(1) $\bar{q}(s) = b(y-ks)/r(s)v$

(2) $\bar{z}(s) = a(y-ks)/pv$

where $v = a + b$

The optimal utility level attainable by households residing at distance s from the CBD, $\bar{u}(s)$, is

(3) $\bar{u}(s) = u(\bar{z},\bar{q},s) = H(b/r(s)v)^b (y-ks)^v e^{-cs}$

where $H = (a/pv)^a$

Assume that the households relocate within their city whenever they can increase their optimal utility level. Then a state of within city spatial equilibrium in which no household is motivated to change residential location will exist whenever the optimal utility level of households $\bar{u}(s)$ is spatially invariant. Namely for

(4) $\bar{u}(s) = \bar{\bar{u}} \qquad s \geq 0$

Equation (4) can be solved to obtain the spatial equilibrium land value distance function $\bar{\bar{r}}(s)$. $\bar{\bar{r}}(s)$ is the functional relationship between land values and distance from the CBD insuring that the optimal utility level attainable by the household be equal to a constant $\bar{\bar{u}}$ irrespective of the households' place of residence. To this effect the equilibrium land values $\bar{\bar{r}}(s)$ are higher in more central areas in order to compensate for the preference of the households for central locations.

$$\bar{\bar{r}}(s) = (b/v)(H/\bar{\bar{u}})(y-ks)^v e^{-cs})^{1/b}$$

The quantity of land $\bar{\bar{q}}(s)$ used by a household at distance s from the CBD when a spatial equilibrium exists can be obtained by replacing $\bar{\bar{r}}(s)$ for r(s) in equation (1).

$$\bar{\bar{q}}(s) = ((\bar{\bar{u}}/H)(y-ks)^{-a}e^{cs})^{1/b}$$

By dividing a unit of land by the spatial equilibrium quantity of land consumed by a household, $\bar{\bar{q}}(s)$, the equilibrium number of households per unit of land is obtained. Hence, the spatial equilibrium population density $\bar{\bar{D}}(s)$ is equal to $(\bar{\bar{q}}(s))^{-1}$ times the number of persons per household m. Namely

$$\bar{\bar{D}}(s) = m((H/\bar{\bar{u}})(y-ks)^{a}e^{-cs})^{1/b}$$

Assume that the spatial equilibrium radius of the urban area $\bar{\bar{s}}$ is identified by the distance in correspondence to which the equilibrium population density equals a given threshold L, $\bar{\bar{D}}(\bar{\bar{s}}) = L$, and that the land in residential use occupies a fixed proportion n of the urban land. Then the equilibrium population of the city $\bar{\bar{P}}$ is defined by the following equation

$$\bar{\bar{P}} = 2\pi n \int_0^{\bar{\bar{s}}} s\bar{\bar{D}}(s)ds$$

or

(5) $$\bar{\bar{P}} = f(y,\bar{\bar{u}}) = W \int_0^{\bar{\bar{s}}} s((y-ks)^{a}e^{-cs})^{1/b}ds$$

where $W = 2\pi nm(H/\bar{\bar{u}})^{1/b}$

It can be easily shown the spatial equilibrium population of a city is an increasing function of y and a decreasing function of $\bar{\bar{u}}$,

$$\frac{\partial\bar{\bar{P}}}{\partial y} = \frac{\partial f}{\partial y} > 0$$

and

$$\frac{\partial\bar{\bar{P}}}{\partial\bar{\bar{u}}} = \frac{\partial f}{\partial\bar{\bar{u}}} < 0$$

Assume that households will migrate from one city of the system into another if they can thereby increase their optimal utility level. Then, an intracity spatial equilibrium exists if the optimal utility level attainable by the households is the same in all the cities of the system. Let this be the case, and assume that the cities in the system are sufficiently spaced so that no two urban areas will overlap. Under these conditions the spatial equilibrium population is identical for all the cities.

Equation (5) expresses the spatial equilibrium population of each city as function of income and of spatial equilibrium utility levels. The relationship among $\bar{\bar{P}}$, y, and $\bar{\bar{u}}$ it specifies, holds for any positive values of these variables. A second relationship between P and y is generated by the initial assumption that the income of the households in a center at first increases and then decreases as the size of the center is increased. Namely

(6) $\quad y = y(P)$

(7) $\quad \frac{dy}{dP} \gtreqless 0 \quad \text{for } P \lesseqgtr P^*$

The assumption is suggested by the following considerations: Since the households in the model are identical, and a constant fraction of the people in each household works "downtown" a given population of a city determines a corresponding amount of labour supply. Since the only income the households in the model perceive is remuneration for work in the CBD, y depends upon the wage rate. Suppose that the wage rate is proportional to the marginal productivity of labour, and that the marginal productivity of labour increases to a maximum as the labour is increased and then declines [3]. The initial increases would depend upon economies of scale and external economies, while the subsequent decrease is due to the diseconomies caused by a too large concentration of economic units. Then the households' incomes are a function of their city's size of the kind indicated by equations (6) and (7).

Assume that for each $\bar{\bar{u}} > 0$ values $\check{P}$ and $\check{y}$ of P and y exist that satisfy simultaneously equations (5) and (6), so that

(8) $\quad y(\check{P}) = \check{y}$

(9) $\quad f(\check{y},\bar{\bar{u}}) = \check{P}$

In other words given $\bar{\bar{u}}$, $\check{P}$ is simultaneously a spatial equilibrium population size corresponding to an income $\check{y}$ and the urban population that generate a marginal productivity of labour corresponding to $\check{y}$. For a given $\bar{\bar{u}}$, $\check{P}$ or $\check{y}$ are determined. However equations (8) and (9) also define 'implicitly' $\bar{\bar{u}}$ as a function of $\check{P}$. Namely

(10) $\quad \bar{\bar{u}} = \bar{\bar{u}}(\check{P})$

The main object of this paper is discussing existence conditions of the value $\overset{*}{P}$ of $\check{P}$ that maximizes $\bar{\bar{u}}$. This value is the optimal population size of the centers so that, if the centers have population size of $\overset{*}{P}$, the optimal utility level of all the households in the system will be at its maximum level. From equations (8), (9), and (10) it follows

(11) $\quad \check{P} = f(y(\check{P}),\bar{\bar{u}}(\check{P}))$

By taking the derivative with respect to $\check{P}$ of both sides of equation (11) we obtain

$$\frac{d\check{P}}{d\check{P}} = 1 = \frac{\partial f}{\partial y}\frac{dy}{d\check{P}} + \frac{\partial f}{\partial \bar{\bar{u}}}\frac{d\bar{\bar{u}}}{d\check{P}}$$

Hence

(12) $$\frac{d\bar{\bar{u}}}{d\check{P}} = \left(1 - \frac{\partial f}{\partial y}\frac{dy}{d\check{P}}\right) / \left(\frac{\partial f}{\partial \bar{\bar{u}}}\right)$$

Assume that $d\bar{u}/d\check{P}$ is continuous for $\check{P}$ greater than zero. Two alternatives are possible. If

$$\frac{d\bar{u}}{d\check{P}} < 0; \ \check{P} > 0$$

the optimum population size of the centers is zero so that the optimal utility levels of the households are the smaller, the larger the center. In this event the existence of urban centers would have to be justified in extraeconomic terms, for instance on military or political grounds. If instead $d\bar{u}/d\check{P}$ is positive for small values of $\check{P}$ and negative for large ones, then at least one value of $\check{P}$ involving a relative maximum of $\bar{u}$ must exist. Therefore let us investigate the sign of $d\bar{u}/d\check{P}$. Since $\partial f/\partial\bar{\bar{u}} < 0$ equation (12) implies that

$$\frac{d\bar{\bar{u}}}{d\check{P}} \gtreqless 0$$

if

$$1 - \frac{\partial f}{\partial y}\frac{dy}{d\check{P}} \lesseqgtr 0$$

namely, if

(13) $$\frac{\partial f}{\partial y}\frac{dy}{d\check{P}} \gtreqless 1$$

$\partial f/\partial y$ is greater than zero. Therefore, it will depend upon $dy/d\check{P}$ whether the equality or the inequalities and, in case, which one of the inequalities of equation (13) will hold. Since $dy/d\check{P}$ is negative for large values of $\check{P}$ and positive for small $\check{P}$'s, the crucial question is whether the parameters of $f(y,\bar{u})$ and of $y(P)$ are such that

$$\frac{dy}{d\check{P}} = \left[\frac{\partial f}{\partial y}\right]^{-1}$$

for some $\check{P} > 0$. If this is the case at least one positive optimum population size $\check{P}$ exists. If $\check{P}$ is given the optimum number of centers $\check{N}$ is easily obtained from the following:

$$T_{PC} = T_P / \check{N} = \check{P}$$

Conclusion

In this paper, the partition of a finite population into centers of optimal size was investigated within an abstract context characterized by a homogeneous plane, identical utility maximizing households, and identical functions relating the household incomes to the size of their city of residence. Spatial equilibrium conditions were introduced in order to insure that no household be motivated to relocate between or within centers. The spatial equilibrium size of the centers was assumed to be an optimal one if it maximized the utility levels attainable by the households. A set of sufficient conditions for the existence of non-zero optimal sizes of centers were derived.

REFERENCES

1. Casetti, Emilio. "Equilibrium Land Values and Population Densities in an Urban Setting," Economic Geography 47 (1971), 16-20.

2. Casetti, Emilio. "On the Derivation of Spatial Equilibrium Land Value Functions," Discussion Paper No. 14, Ohio State University, Department of Geography, 1970.

3. Williamson, J. G. and J. A. Swanson. "The Growth of Cities in the American Northeast, 1820-1870," Exploration in Entrepreneurial History, Second Series, 4 (1966), 28-29.